Thrust Equation for a Turbofan
Double Inlet/Outlet

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April 8, 2017

1 Introduction

In this document, we will be deriving the thrust equation for a double inlet/double outlet engine. I use the term "turbofan" in the title as more of a buzz word than anything, because the derived equation will hold for any engine with a double inlet and double outlet.

Mass Flow Rate

Mass flow rates will be used throughout this derivation, so it's important to understand what the term means, as well as the units associated with it. Mass flow rates are useful for evaluating general thrust of engines because it is a number that can be easily found for pretty much any engine on the internet. The mass flow rate of a fluid is defined as the amount mass of fluid flowing per unit time. The equation for mass flow rate is given in Eq. (1).

$$\dot{m} = \rho u A \tag{1}$$

To see how the units work out for this equation, see Eq. (2) below.

$$\left[\frac{kg}{s}\right] = \left[\frac{kg}{m^3}\right]\left[\frac{m}{s}\right]\left[m^2\right] \tag{2}$$

Velocity

In a general three-dimensional flow, we have three velocity components, one for each coordinate direction. We can write the general formula for the velocity at any given point as seen in Eq. (3).

$$\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k} \tag{3}$$

The goal of the analysis is to develop and expression for the thrust, which will be aligned with the X-direction. Thus, we can say that the Y-direction and Z-direction velocities are zero, and write the velocity vector as follows.

$$\vec{V} = u\hat{\imath} \tag{4}$$

Area

One of the most deceivingly confusing parts the conservation equations is the control volume surface area term, and what it means. For a blog post regarding this topic, visit the following website.

`http://www.joshtheengineer.com/2017/01/02/surface-double-integrals/`

To sum it all up, the surface area terms used in the conservation equations are shown below.

$$\begin{aligned} d\vec{S_1} &= \hat{n}_1 dA_1 \\ d\vec{S_2} &= \hat{n}_2 dA_2 \end{aligned} \tag{5}$$

This means that after integration based on the outward normal vectors, we will have the following.

$$\begin{aligned} S_1 &= -A_1 \\ S_2 &= +A_2 \end{aligned} \tag{6}$$

2 Engine Control Volume Schematic

A schematic of the engine and its surrounding control volume can be seen in Fig. 1. The engine is drawn in red to distinguish it from the control volume and associated variables. Flow comes in from the left and exits to the right.

Because we are dealing with a dual inlet and outlet, the flow through the engine is split between a cold flow and a hot flow, denoted by C and H, respectively. The hot flow is the flow that goes through the core of the engine. The cold flow is the flow that bypasses the core of the engine, also called the bypass flow. The flow through the core is called "hot" because it is mixed with fuel and ignited. The bypass flow is called "cold" because it does not go through a combustion process.

For a dual inlet and outlet engine, we will have different mass flow rates for the two streams passing through the engine. The exit mass flow rate for the engine is the sum of the core and bypass mass flow rates.

$$\dot{m}_e = \dot{m}_{e,H} + \dot{m}_{e,C} \tag{7}$$

The exit mass flow rate of the core stream includes the incoming air to the core and the mass flow rate of the fuel.

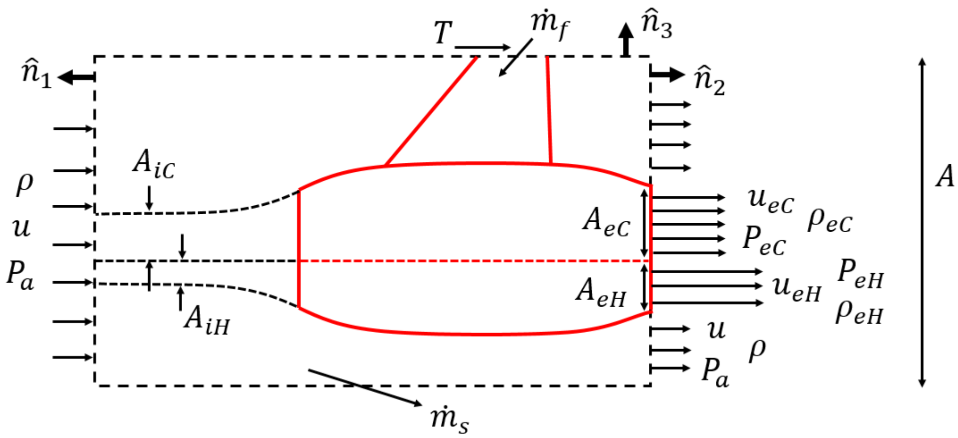

Figure 1: Engine control volume schematic.

$$\dot{m}_{e,H} = \dot{m}_{a,H} + \dot{m}_f \tag{8}$$

The total inlet area of the engine is the sum of the core and bypass inlet areas.

$$A_i = A_{i,H} + A_{i,C} \tag{9}$$

Similarly, the total outlet area of the engine is the sum of the core and bypass outlet areas.

$$A_e = A_{e,H} + A_{e,C} \tag{10}$$

3 Conservation of Mass

Here is the mass conservation in integral form.

$$\iint \rho \vec{V} \cdot d\vec{S} = 0 \tag{11}$$

Because there are more terms to deal with here than in the single inlet/single outlet derivation, we will break them down into individual components to make it easier to follow along. The core stream is still denoted with an "H", and the bypass stream is still denoted with a "C". The flow external to the engine is denoted with an "E". From the mass conservation equation, we can see that we will need to integrate over some areas, shown below.

$$E_{in} + C_{in} + H_{in} + E_{out} + C_{out} + H_{out} + Other = 0 \tag{12}$$

Let's write out each of these terms individually first.

$$
\begin{aligned}
E_{in} &: \rho \left[(u\hat{\imath}) \cdot -\left(A - A_{i,C} - A_{i,H} \right) \hat{\imath} \right] = -\rho u \left(A - A_{i,C} - A_{i,H} \right) = -\rho u \left(A - A_i \right) \\
C_{in} &: \rho \left[(u\hat{\imath}) \cdot (-A_{i,C}\hat{\imath}) \right] = -\rho u A_{i,C} = -\dot{m}_{a,C} \\
H_{in} &: \rho \left[(u\hat{\imath}) \cdot (-A_{i,H}\hat{\imath}) \right] = -\rho u A_{i,H} = -\dot{m}_{a,H} \\
E_{out} &: \rho \left[(u\hat{\imath}) \cdot \left(A - A_{e,C} - A_{e,h} \right) \hat{\imath} \right] = \rho u \left(A - A_{e,C} - A_{e,H} \right) = \rho u \left(A - A_e \right) \\
C_{out} &: \rho_{e,C} \left[(u_{e,C}\hat{\imath}) \cdot (A_{e,C}\hat{\imath}) \right] = \rho_{e,C} u_{e,C} A_{e,C} = \dot{m}_{e,C} \\
H_{out} &: \rho_{e,H} \left[(u_{e,H}\hat{\imath}) \cdot (A_{e,H}\hat{\imath}) \right] = \rho_{e,H} u_{e,H} A_{e,H} = \dot{m}_{e,H} \\
Other &: \dot{m}_s - \dot{m}_f
\end{aligned}
\tag{13}
$$

Using the above expressions, we can write them in the full mass conservation equation.

$$
-\rho u \left(A - A_i \right) - \dot{m}_{a,C} - \dot{m}_{a,H} + \rho u \left(A - A_e \right) + \dot{m}_{e,C} + \dot{m}_{e,H} + \dot{m}_s - \dot{m}_f = 0 \tag{14}
$$

Now we will solve for $\dot{m}_s$ and consolidate terms.

$$
\begin{aligned}
\dot{m}_s &= \dot{m}_{a,C} + \dot{m}_{a,H} - \dot{m}_{e,C} - \dot{m}_{e,H} + \dot{m}_f + \rho u \left(A - A_i \right) - \rho u \left(A - A_e \right) \\
&= \dot{m}_{a,C} + \dot{m}_{a,H} - \dot{m}_{e,C} - \dot{m}_{e,H} + \dot{m}_f + \cancel{\rho u A} - \rho u A_i - \cancel{\rho u A} + \rho u A_e
\end{aligned}
\tag{15}
$$

We end up with the final result below.

$$
\boxed{\dot{m}_s = \dot{m}_{a,C} + \dot{m}_{a,H} - \dot{m}_{e,C} - \dot{m}_{e,H} + \dot{m}_f + \rho u \left(A_e - A_i \right)} \tag{16}
$$

4 Conservation of Momentum

Here is the mass conservation in integral form.

$$
\iint\limits_S \rho \vec{V} \left(\vec{V} \cdot d\vec{S} \right) + \iint\limits_S P d\vec{S} = T \tag{17}
$$

Keeping track of every term in this equation is going to be even more difficult, because for every term we had in the mass conservation equation, we will need two of here due to the two surface integral terms. Let us again keep the same nomenclature scheme (E, C, and H), but denote the first integral term by the subscript 1 and the second integral term by the subscript 2. Below is the thrust equation written in terms of this nomenclature.

$$
\begin{aligned}
&E_{in,1} + C_{in,1} + H_{in,1} + E_{out,1} + C_{out,1} + H_{out,1} \\
&+ E_{in,2} + C_{in,2} + H_{in,2} + E_{out,2} + C_{out,2} + H_{out,2} + Other = T
\end{aligned}
\tag{18}
$$

We can now write each of these terms individually. Let's start with the terms for the first integral only.

$$\begin{aligned}
E_{in,1} &: \rho\,(u\hat{\imath})\left[(u\hat{\imath})\cdot -\left(A - A_{i,C} - A_{i,H}\right)\hat{\imath}\right] = -\rho u^2\left(A - A_i\right) \\
C_{in,1} &: \rho\,(u\hat{\imath})\left[(u\hat{\imath})\cdot\left(-A_{i,C}\hat{\imath}\right)\right] = -\rho u A_{i,C} u = -\dot{m}_{a,C} u \\
H_{in,1} &: \rho\,(u\hat{\imath})\left[(u\hat{\imath})\cdot\left(-A_{i,H}\hat{\imath}\right)\right] = -\rho u A_{i,H} u = -\dot{m}_{a,H} u \\
E_{out,1} &: \rho\,(u\hat{\imath})\left[(u\hat{\imath})\cdot\left(A - A_{e,C} - A_{e,H}\right)\hat{\imath}\right] = \rho u^2\left(A - A_e\right) \\
C_{out,1} &: \rho_{e,C}\left(u_{e,C}\hat{\imath}\right)\left[\left(u_{e,C}\hat{\imath}\right)\cdot\left(A_{e,C}\hat{\imath}\right)\right] = \rho_{e,C} u_{e,C} A_{e,C}\left(u_{e,C}\right) = \dot{m}_{e,C} u_{e,C} \\
H_{out,1} &: \rho_{e,H}\left(u_{e,H}\hat{\imath}\right)\left[\left(u_{e,H}\hat{\imath}\right)\cdot\left(A_{e,H}\hat{\imath}\right)\right] = \rho_{e,H} u_{e,H} A_{e,H}\left(u_{e,H}\right) = \dot{m}_{e,H} u_{e,H}
\end{aligned} \tag{19}$$

Now let's go through the terms associated with the second integral.

$$\begin{aligned}
E_{in,2} &: P_a\left[-\left(A - A_{i,C} - A_{i,H}\right)\right] = -P_a\left(A - A_{i,C} - A_{i,H}\right) \\
C_{in,2} &: P_a\left(-A_{i,C}\right) = -P_a A_{i,C} \\
H_{in,2} &: P_a\left(-A_{i,H}\right) = -P_a A_{i,H} \\
E_{out,2} &: P_a\left[\left(A - A_{e,C} - A_{e,H}\right)\right] = P_a\left(A - A_{e,C} - A_{e,H}\right) \\
C_{out,2} &: P_{e,C}\left(A_{e,C}\right) = P_{e,C} A_{e,C} \\
H_{out,2} &: P_{e,H}\left(A_{e,H}\right) = P_{e,H} A_{e,H}
\end{aligned} \tag{20}$$

The last thing to take into account is the momentum crossing the control surface due to the $\dot{m}_s$. This term is positive because we have drawn the vector going out of the control volume, which is in the same direction as the outward normal. Because the fuel flow rate is small, we are neglecting the momentum crossing the control surface due to the fuel flow.

$$Other: \;\dot{m}_s u \tag{21}$$

Now we can combine all the terms in the momentum conservation equation.

$$\begin{aligned}
-\rho u^2\left(A - A_i\right) &- \dot{m}_{a,C} u - \dot{m}_{a,H} u + \rho u^2\left(A - A_e\right) \\
&+ \dot{m}_{e,C} u_{e,C} + \dot{m}_{e,H} u_{e,H} + \dot{m}_s u \\
&- P_a\left(A - A_{i,C} - A_{i,H}\right) - P_a A_{i,C} - P_a A_{i,H} \\
&+ P_a\left(A - A_{e,C} - A_{e,H}\right) + P_{e,C} A_{e,C} + P_{e,H} A_{e,H} = T
\end{aligned} \tag{22}$$

In the next step, we will both combine the momentum-area terms, and expand out the pressure terms.

$$\begin{aligned}
&\rho u^2\left(A_i - A_e\right) - \dot{m}_{a,C} u - \dot{m}_{a,H} u + \dot{m}_{e,C} u_{e,C} + \dot{m}_{e,H} u_{e,H} + \dot{m}_s u \\
&\cancel{-P_a A} + \cancel{P_a A_{i,C}} + \cancel{P_a A_{i,H}} - \cancel{P_a A_{i,C}} - \cancel{P_a A_{i,H}} + \cancel{P_a A} \\
&- P_a A_{e,C} - P_a A_{e,H} + P_{e,C} A_{e,C} + P_{e,H} A_{e,H} = T
\end{aligned} \tag{23}$$

We can simplify this equation and solve for the thrust, T.

$$\boxed{\begin{aligned}
T = \rho u^2\left(A_i - A_e\right) &- \dot{m}_{a,C} u - \dot{m}_{a,H} u + \dot{m}_{e,C} u_{e,C} + \dot{m}_{e,H} u_{e,H} + \dot{m}_s u \\
&+ A_{e,C}\left(P_{e,C} - P_a\right) + A_{e,H}\left(P_{e,H} - P_a\right)
\end{aligned}} \tag{24}$$

5 Combining Mass and Momentum

Recall the two boxed equations from above. We will plug Eq. (16) into Eq. (24) to get the following expression.

$$\begin{aligned} T =&\rho u^2 (A_i - A_e) - \dot{m}_{a,C} u - \dot{m}_{a,H} u + \dot{m}_{e,C} u_{e,C} + \dot{m}_{e,H} u_{e,H} \\ & u \left[\dot{m}_{a,C} + \dot{m}_{a,H} - \dot{m}_{e,C} - \dot{m}_{e,H} + \dot{m}_f + \rho u (A_e - A_i)\right] \\ & + A_{e,C} (P_{e,C} - P_a) + A_{e,H} (P_{e,H} - P_a) \end{aligned} \tag{25}$$

Distribute the velocity u through in the third to last term, and cancel like values.

$$\begin{aligned} T = & \cancel{\rho u^2 (A_i - A_e)} - \cancel{\dot{m}_{a,C} u} - \cancel{\dot{m}_{a,H} u} + \dot{m}_{e,C} u_{e,C} + \dot{m}_{e,H} u_{e,H} + \\ & \cancel{\dot{m}_{a,C} u} + \cancel{\dot{m}_{a,H} u} - \dot{m}_{e,C} u - \dot{m}_{e,H} u + \dot{m}_f u + \cancel{\rho u^2 (A_e - A_i)} \\ & A_{e,C} (P_{e,C} - P_a) + A_{e,H} (P_{e,H} - P_a) \end{aligned} \tag{26}$$

After canceling the like terms in the above equation and grouping the terms with velocity u, we are left with the following equation.

$$\begin{aligned} T = & \dot{m}_{e,C} u_{e,C} + \dot{m}_{e,H} u_{e,H} + u \left[\dot{m}_f - \dot{m}_{e,C} - \dot{m}_{e,H}\right] \\ & + A_{e,C} (P_{e,C} - P_a) + A_{e,H} (P_{e,H} - P_a) \end{aligned} \tag{27}$$

To continue further, we need to note that the cold exit mass flow rate is equal to the cold inlet mass flow rate (Eq. (28)), the hot exit mass flow rate is equal to the hot inlet mass flow rate plus the fuel mass flow rate (Eq. (29)), and the fuel-to-air ratio is equal to the mass flow rate of the fuel over the hot mass flow rate (Eq. (30)).

$$\dot{m}_{e,C} = \dot{m}_{a,C} \tag{28}$$

$$\dot{m}_{e,H} = \dot{m}_{a,H} + \dot{m}_f \tag{29}$$

$$f = \frac{\dot{m}_f}{\dot{m}_{a,H}} \tag{30}$$

Using the above equations, we can simplify our thrust expression.

$$\begin{aligned} T = & \dot{m}_{a,C} u_{e,C} + (\dot{m}_{a,H} + \dot{m}_f) u_{e,H} + u \left[\cancel{\dot{m}_f} - \dot{m}_{a,C} - (\dot{m}_{a,H} + \cancel{\dot{m}_f})\right] \\ & + A_{e,C} (P_{e,C} - P_a) + A_{e,H} (P_{e,H} - P_a) \end{aligned} \tag{31}$$

Move some of these terms around to get like terms consolidated.

$$\begin{aligned} T = & \ (\dot{m}_{a,H} + \dot{m}_f) u_{e,H} - \dot{m}_{a,H} u - \dot{m}_{a,C} u + \dot{m}_{a,C} u_{e,C} \\ & + A_{e,C} (P_{e,C} - P_a) + A_{e,H} (P_{e,H} - P_a) \end{aligned} \tag{32}$$

Now we can multiply the first term in the equation by $\dot{m}_{a,H}/\dot{m}_{a,H}$, and combine the third and fourth terms by factoring out the $\dot{m}_{a,C}$ from each.

$$\begin{aligned} T = \dot{m}_{a,H} u_{e,H} (1+f) - \dot{m}_{a,H} u + \dot{m}_{a,C} (u_{e,C} - u) \\ + A_{e,C} (P_{e,C} - P_a) + A_{e,H} (P_{e,H} - P_a) \end{aligned} \tag{33}$$

The last step is to factor out the $\dot{m}_{a,H}$ from the first and second terms, and we are left with our dual inlet/outlet thrust equation.

$$\boxed{\begin{aligned} T = \dot{m}_{a,H} [u_{e,H} (1+f) - u] + \dot{m}_{a,C} (u_{e,C} - u) \\ + A_{e,C} (P_{e,C} - P_a) + A_{e,H} (P_{e,H} - P_a) \end{aligned}} \tag{34}$$

6 Thrust Equation Simplifications

If the nozzle is not choked, then the exit pressure of the engine will equal the atmospheric pressure. The topic of nozzle choking will be discussing in a different post. If the nozzle is not choked, then the second term in the thrust equation disappears because $P_e = P_a$.

$$\boxed{T = \dot{m}_{a,H} [u_{e,H} (1+f) - u] + \dot{m}_{a,C} (u_{e,C} - u)} \tag{35}$$

In typical engines, the fuel-to-air ratio, f, is very small. If we assume that $f \ll 1$, then we can write the thrust equation as follows. This is still assuming that our nozzle is not choked.

$$\boxed{T = \dot{m}_{a,H} (u_{e,H} - u) + \dot{m}_{a,C} (u_{e,C} - u)} \tag{36}$$

From all these thrust equations, we can see that both the core and bypass flows contribute to the overall thrust. Increasing the exit velocity of both the core and bypass will increase thrust, while increasing flight velocity, u, will decrease thrust.